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Controlled Dense Coding between Multi-Parties

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Abstract Controlled dense coding via multi-particles GHZ state and multi-particles GHZ-class state are exploited in this letter. The quantum channel and the amount of information between the senders and the receivers are controlled by the supervisor via his local measurement. The amount of information is determined by Charlie's measurement in the former case of GHZ state, and also by the coefficients of the original GHZ-class state in the latter case.

Keywords Controlled dense coding · GHZ-class state · Amount of information

1 Introduction

Quantum dense coding or superdense coding is one of the important branches of quantum information theory [1]. It has been widely studied both theoretically and experimentally [2–5]. The basic idea of quantum dense coding is that quantum mechanics allows one to encode information in the quantum states that is more dense than classical coding. In 2001, Hao et al. present a controlled dense coding scheme using the GHZ state [6]. In this case the sender (Alice) can send information to the receiver (Bob). The quantum channel between Alice and Bob and the information Alice send to Bob are controlled by supervisor (Charlie), whose local measurement serves as quantum erasure. This scheme has been realized in experiments [7, 8]. Recently, Chen and Kuang have generalized the controlled dense coding protocol of the three-particle GHZ quantum channel to the case of an $(N+2)$-particle GHZ quantum channel via a series of local measurements [9]. Fu et al. studied controlled quantum dense coding in a four-particle non-maximal quantum channel via local measurements [10]. In this letter, we present a controlled dense coding scheme between multi-parties. To present our scheme clear, we first present four-parties controlled dense coding scheme via a four-qubits GHZ state, and then the general scheme for controlled dense coding between multi-parties via multi-qubits GHZ state, finally, we consider multi-parties sharing a multi-qubits GHZ-class state.

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2 Four-Parties Sharing a GHZ State

Suppose the four qubits 1, 2, 3 and 4 held by sender 1 (Alice), sender 2, receiver (Bob) and the supervisor (Charlie), respectively, are in the four-qubits GHZ state

$$|\psi\rangle_{1234} = \frac{1}{\sqrt{2}}(|0000\rangle_{1234} + |1111\rangle_{1234}). \tag{1}$$

First Charlie measures his qubit under the basis $\{|+\rangle_4, |-\rangle_4\}$ and sends the result to Alice through a classical channel, where

$$|+\rangle_4 = \cos\theta|0\rangle_4 + \sin\theta|1\rangle_4, \qquad |-\rangle_4 = \sin\theta|0\rangle_4 - \cos\theta|1\rangle_4, \tag{2}$$

and $0 \le \theta \le \pi/4$. Rewriting the four-qubits GHZ state in the new basis gives

$$|\psi\rangle_{1234} = \frac{1}{\sqrt{2}}(|\phi\rangle_{123} \otimes |+\rangle_4 + |\varphi\rangle_{123} \otimes |-\rangle_4), \tag{3}$$

where

$$|\phi\rangle_{123} = \cos\theta|000\rangle_{123} + \sin\theta|111\rangle_{123}, \qquad |\varphi\rangle_{123} = \sin\theta|000\rangle_{123} - \cos\theta|111\rangle_{123}. \tag{4}$$

Obviously, Charlie's measurement gives two results $|+\rangle_4$ and $|-\rangle_4$ with the same probability $\frac{1}{2}$. If the measurement result is $|+\rangle_4$ or $|-\rangle_4$, the state of qubits 1, 2 and 3 collapses to $|\phi\rangle_{123}$ or $|\varphi\rangle_{123}$ respectively.

Now we consider the former case. Generally speaking, $|\phi\rangle_{123}$ is not maximally entangled state and the success probability of dense coding with it is less than 1. After receiving the measurement result, Alice introduces an auxiliary qubit with original state $|0\rangle_{aux}$ and performs the unitary transformation

$$U_1 = \begin{pmatrix} \tan\theta & 0 & \sqrt{1-\tan^2\theta} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\tan^2\theta} & 0 & -\tan\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

on her qubit and the auxiliary qubit under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$. The collective unitary operator $U_1 \otimes I_4$ (I_4 being identity transformation on qubits 2 and 3) transforms the state $|0\rangle_{aux} \otimes |\phi\rangle_{123}$ to

$$|\psi\rangle_{aux123} = \sqrt{2}\sin\theta|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|000\rangle_{123} + |111\rangle_{123})\right] + \sqrt{\cos 2\theta}|1\rangle_{aux} \otimes |000\rangle_{123}. \tag{6}$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If Alice gets the result $|1\rangle_{aux}$, qubits 1, 2 and are unentangled. In this case 1 bit of information is transmitted through channel. If she gets $|0\rangle_{aux}$, the state of qubits 1, 2 and 3 collapses to a maximally entangled state and she can perform dense coding. Alice can perform any of the four transformations $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her qubit and sends it to Bob. However, if so does the second sender, Bob can not identify the transformations performed by Alice and the second sender uniquely. Thus not all collective transformations are allowed, and some restriction has to be made [11]. In the present case, this problem can be solved by allowing Alice to perform all four possible unitary transformations $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$, and the second sender to perform transformations $\{I, \sigma_X\}$ or $\{i\sigma_Y, \sigma_Z\}$ [12]. When Bob receives the transformed qubits from Alice and the second sender, he can find what information they have encoded by just one measurement on qubits 1, 2 and 3. That it is to say, Bob gets 3 bits information with a single measurement. So on average, the amount of transmitted information is $1+\sin^2\theta$ bits, where the probability $\frac{1}{2}$ of Charlie obtaining $|+\rangle_4$ is considered.

For the case of Charlie's result being $|-\rangle_4$, the procedure is similar and the amount of transmitted information is the same. So totally

$$I = 2 + 2\sin^2\theta \tag{7}$$

bits of information are transmitted from the senders to receiver. From (7) we can see that by adjusting the value of θ Charlie can control the entanglement between the senders and receiver, and hence transmission between them.

3 Multi-Parties Sharing a GHZ State

In this section, we generalize the scheme to multi-parties. Suppose N senders (the first one being Alice), receiver Bob and supervisor Charlie share a $(N+2)$-qubits GHZ state

$$|\psi\rangle_{12\cdots N+2} = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle_{12\cdots N+2} + |11\cdots 1\rangle_{12\cdots N+2}), \tag{8}$$

where the former N qubits, qubit $N+1$, qubit $N+2$ belongs to N senders, Bob and Charlie, respectively.

First, Charlie measures his qubit under the basis

$$|+\rangle_{N+2} = \cos\theta|0\rangle_{N+2} + \sin\theta|1\rangle_{N+2}, \qquad |-\rangle_{N+2} = \sin\theta|0\rangle_{N+2} - \cos\theta|1\rangle_{N+2}. \tag{9}$$

Rewriting the multi-qubits GHZ state in the new basis $\{|+\rangle_{N+2}, |-\rangle_{N+2}\}$ gives

$$|\psi\rangle_{12\cdots N+2} = \frac{1}{\sqrt{2}}(|\phi\rangle_{12\cdots N+1}\otimes|+\rangle_{N+2} + |\varphi\rangle_{12\cdots N+1}\otimes|-\rangle_{N+2}), \tag{10}$$

where

$$\begin{aligned} |\phi\rangle_{12\cdots N+1} &= \cos\theta|00\cdots 0\rangle_{12\cdots N+1} + \sin\theta|11\cdots 1\rangle_{12\cdots N+1}, \\ |\varphi\rangle_{12\cdots N+1} &= \sin\theta|00\cdots 0\rangle_{12\cdots N+1} - \cos\theta|11\cdots 1\rangle_{12\cdots N+1}. \end{aligned} \tag{11}$$

From (10) we can see that the von Neumann measurement of qubit $N+2$ by Charlie gives the readout $|+\rangle_{N+2}$ or $|-\rangle_{N+2}$ with equal probability $\frac{1}{2}$. Now we consider only the case in which Charlie's measurement result is $|+\rangle_{N+2}$ and the state of qubits $1, 2, \ldots, N+1$ collapses to $|\phi\rangle_{12\cdots N+1}$.

After receiving the measurement result, Alice introduces an auxiliary qubit with original state $|0\rangle_{aux}$ and performs the unitary transformation (5) on her qubit and the auxiliary qubit under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$. This collective unitary operator $U_1 \otimes I_{2^{N-1}}$ transforms the state $|0\rangle_{aux}\otimes|\phi\rangle_{12\cdots N+1}$ to

$$\begin{aligned} |\psi\rangle_{aux12\cdots N+1} = {} & \sqrt{2}\sin\theta|0\rangle_{aux}\otimes\left[\frac{1}{\sqrt{2}}(|00\cdots 0\rangle_{12\cdots N+1} + |11\cdots 1\rangle_{12\cdots N+1})\right] \\ & + \sqrt{\cos 2\theta}|1\rangle_{aux}\otimes|00\cdots 0\rangle_{12\cdots N+1}. \end{aligned} \tag{12}$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If she gets $|1\rangle_{aux}$, qubits $1, 2, \ldots,$ and $N+1$ are unentangled and N bits of information is transmitted through channel. If she gets $|0\rangle_{aux}$, the state of qubits $1, 2, \ldots,$ and $N+1$ collapses to a maximally entangled state. Now let Alice performs any one of the four transformations $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her qubit, and the sender $2, \ldots,$ and N perform only transformations $\{I, \sigma_X\}$ or $\{i\sigma_Y, \sigma_Z\}$. By this way the N senders can encode their messages by performing their allowed transformations to the qubits at their proposal and send the qubits to Bob. After receiving these qubits, Bob can perform a single measurement on all his $N+1$ qubits so that he can reads out the messages the N senders have encoded. In this way, Bob can get $N+1$ bits of information by only one measurement.

The case of Charlie's result being $|-\rangle_{N+2}$ can be treated in a similar way and the amount of transmitted information is the same. So on average

$$I' = N + 2\sin^2\theta \tag{13}$$

bits of information are transmitted from N senders to Bob.

4 Multi-Parties Sharing a GHZ-Class State

Maximal entangled states are very difficult to prepare and store in practical applications, so it is important to consider the performance of dense coding when the states shared between the senders and the receiver are partially entangled [8]. In this section, we suppose the channel is a GHZ-class state

$$|\psi\rangle_{12\cdots N+2} = \alpha|00\cdots 0\rangle_{12\cdots N+2} + \beta|11\cdots 1\rangle_{12\cdots N+2}, \tag{14}$$

where $\alpha^2 + \beta^2 = 1$ and $0 < \alpha \le \beta$. As before, the former N qubits, qubit $N+1$ and qubit $N+2$ belongs to the N senders (the first one being Alice), Bob and Charlie, respectively.

In order to control the entanglement between the senders and the receiver, Charlie measures his qubit under the same basis $\{|+\rangle_{N+2}, |-\rangle_{N+2}\}$ and sends the measurement result to Alice. Because the multi-qubits GHZ state can be rewritten as

$$|\psi\rangle_{12\cdots N+2} = |\xi\rangle_{12\cdots N+1} \otimes |+\rangle_{N+2} + |\zeta\rangle_{12\cdots N+1} \otimes |-\rangle_{N+2}, \tag{15}$$

where

$$\begin{aligned} |\xi\rangle_{12\cdots N+1} &= \alpha\cos\theta|00\cdots 0\rangle_{12\cdots N+1} + \beta\sin\theta|11\cdots 1\rangle_{12\cdots N+1}, \\ |\zeta\rangle_{12\cdots N+1} &= \alpha\sin\theta|00\cdots 0\rangle_{12\cdots N+1} - \beta\cos\theta|11\cdots 1\rangle_{12\cdots N+1} \end{aligned} \tag{16}$$

are not normalized. Charlie can obtain two probable results as before, but later operations are no longer similar.

We first consider the case in which Charlie's result is $|+\rangle_{N+2}$ and the state of qubits $1, 2, \ldots, N+1$ collapses to $|\xi\rangle_{12\cdots N+1}$. After receiving the measurement result, Alice introduces an auxiliary qubit with state $|0\rangle_{aux}$ and performs the unitary transformation

$$U_2 = \begin{pmatrix} \beta\tan\theta/\alpha & 0 & \sqrt{1-\beta^2\tan^2\theta/\alpha^2} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\beta^2\tan^2\theta/\alpha^2} & 0 & -\beta\tan\theta/\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

on her qubit and the auxiliary qubit under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$. The collective unitary operator $U_2 \otimes I_{2^{N-1}}$ transforms the state $|0\rangle_{aux} \otimes |\xi\rangle_{12\cdots N+1}$ to

$$|\psi\rangle_{aux12\cdots N+1} = \sqrt{2}\beta\sin\theta|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|00\cdots 0\rangle_{12\cdots N+1} + |11\cdots 1\rangle_{12\cdots N+1})\right] + \sqrt{\alpha^2\cos^2\theta - \beta^2\sin^2\theta}|1\rangle_{aux} \otimes |00\cdots 0\rangle_{12\cdots N+1}. \quad (18)$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If she gets $|1\rangle_{aux}$, qubits $1, 2, \ldots, N+1$ are unentangled and at most N bits of information can be transmitted through the channel. If she get $|0\rangle_{aux}$, the state of qubits $1, 2, \ldots, N+1$ is maximally entangled and Bob can get $N+1$ bits of information by only one measurement. So on average

$$I_1 = (N+2)\beta^2\sin^2\theta + N\alpha^2\cos^2\theta \quad (19)$$

bits of information are transmitted from N senders to Bob.

If Charlie's measurement result is $|-\rangle_{N+2}$, the state of qubits $1, 2, \ldots, N+1$ collapses to $|\zeta\rangle_{12\cdots N+1}$. This situation is more complicated because the relative size of coefficient $\alpha\sin\theta$ to $\beta\cos\theta$ in $|\zeta\rangle_{12\cdots N+1}$ is no longer definite.

In order to purify the state of $|\zeta\rangle_{12\cdots N+1}$, Alice can introduces an auxiliary qutrit (quantum system with 3-dimensional Hilbert space) with original state $|0\rangle_{aux}$ and performs an unitary transformation

$$U_2' = \begin{pmatrix} \beta/\alpha & 0 & \sqrt{1-\beta^2/\alpha^2} & 0 & 0 & 0 \\ 0 & -\tan\theta & 0 & \sqrt{1-\tan^2\theta} & 0 & 0 \\ \sqrt{1-\beta^2/\alpha^2} & 0 & -\beta/\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \sqrt{1-\tan^2\theta} & 0 & \tan\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (20)$$

on the auxiliary qutrit and qubit 1 under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1, |2\rangle_{aux}|0\rangle_1, |2\rangle_{aux}|1\rangle_1\}$. The collective unitary transformation $U_2' \otimes I_{2^{N-1}}$ transforms the state $|0\rangle_{aux} \otimes |\zeta\rangle_{12\cdots N+1}$ to

$$|\zeta\rangle_{aux12\cdots N+1} = \sqrt{2}\beta\sin\theta|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|00\cdots 0\rangle_{12\cdots N+1} + |11\cdots 1\rangle_{12\cdots N+1})\right] + \sin\theta\sqrt{\alpha^2-\beta^2}|1\rangle_{aux} \otimes |00\cdots 0\rangle_{12\cdots N+1} - \beta\sqrt{\cos 2\theta}|2\rangle_{aux} \otimes |01\cdots 1\rangle_{12\cdots N+1}. \quad (21)$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}, |2\rangle_{aux}\}$. If Alice gets the result $|1\rangle_{aux}$ or $|2\rangle_{aux}$, the state of qubits $1, 2, \ldots, N+1$ is unentangled, and N bits of information is transmitted through it for each case. If she gets $|0\rangle_{aux}$, the state of qubits $1, 2, \ldots, N+1$ is maximally entangled, and $N+1$ bits of information are transmitted. So the senders can transmit

$$I_2 = (N\alpha^2 + 2\beta^2)\sin^2\theta + N\beta^2\cos^2\theta \quad (22)$$

bits of information on average. Thus the average amount of information transmitted from Alice to Bob adds up to

$$I'' = I_1 + I_2 = N + 4\beta^2 \sin^2\theta. \tag{23}$$

We can see that, when multi-parties share a GHZ-class state, things become more complicated. Corresponding to different measurement results of Charlie, Alice must introduce different kinds of quantum system as auxiliary particle to purify the state of $|\xi\rangle_{12\cdots N+1}$ or $|\zeta\rangle_{12\cdots N+1}$ and there is no more similarity. Moreover, the communication capacity I'' depends not only on the measurement angle θ but also on the coefficients of the original GHZ-class state, which is different from that in the former case.

5 Summary

In summary, we give a scheme of realizing dense coding via multi-particles GHZ state and multi-particles GHZ-class state. We can see that Charlie can control the transmitted classical amount of information by adjusting local measurement angle. In the former case, the amount of information between the senders and the receiver (Bob) are determined by Charlie's measurement. But in the latter case, the amount of information is determined by both Charlie's measurement and the coefficients of the original multi-particles GHZ-class state, and subsequent procedures after Charlie's two probable results are no longer parallel.

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